Lecture 9. Introduction: Second-Order Linear Equations

1. Review: Definition of second-order linear equations

Recall a *linear second-order equation* can be written in the form

$$A(x)y'' + B(x)y' + C(x)y = F(x) \tag{1}$$

We assume that $A(x), B(x), C(x)$ and $F(x)$ are continuous functions on some open interval I.

For example,

$$e^x y'' + (\cos x)y' + (1+\sqrt{x})y = \tan^{-1} x$$

is linear because the dependent variable y and its derivatives y' and y'' appear linearly.

The equations

$$y'' = yy' \quad \text{and} \quad y'' + 2(y')^2 + 4y^3 = 0$$

are **not** linear because products and powers of y or its derivatives appear.

2. Homogeneous Second-Order Linear Equations

If the function $F(x) = 0$ on the right-hand side of Eq. (1), then we call Eq. (1) a **homogeneous** linear equation; otherwise, it is **nonhomogeneous**. In general, the homogeneous linear equation associated with Eq. (1) is

$$A(x)y'' + B(x)y' + C(x)y = 0 \tag{2}$$

For example, the second-order equation

$$2x^2y'' + 2xy' + 3y = \sin x$$

is nonhomogeneous; its associated homogeneous equation is

$$2x^2y'' + 2xy' + 3y = 0$$

Consider

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

Assume that $A(x) \neq 0$ at each point of the open interval I, we can divide each term in Eq. (1) by $A(x)$ and write it in the form

$$y'' + p(x)y' + q(x)y = f(x)$$

We will discuss first the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{3}$$

Recall the Eq (3)

$$y'' + p(x)y' + q(x)y = 0 \tag{3}$$

Theorem 1 Principle of Superposition for Homogeneous Equations

Let y_1 and y_2 be two solutions of the homogeneous linear equation in Eq. (3) on the interval I. If c_1 and c_2 are constants, then the linear combination

$$y = c_1 y_1 + c_2 y_2$$

is also a solution of Eq. (3) on I.

Idea of the proof:

Since y_1 and y_2 are both solutions to Eq(3), we have

$y_1'' + p(x)y_1' + q(x)y_1 = 0$ and $y_2'' + p(x)y_2' + q(x)y_2 = 0$

Multiply the equtions by c_1 and c_2, respectively, we have

$c_1 y_1'' + p(x)c_1 y_1' + q(x)c_1 y_1 = 0$ and $c_2 y_2'' + p(x)c_2 y_2' + q(x)c_2 y_2 = 0$

Add the two equations above together, we have

$$(c_1 y_1 + c_2 y_2)'' + p(x)(c_1 y_1 + c_2 y_2)' + q(x)(c_1 y_1 + c_2 y_2) = 0$$

Therefore, $y = c_1 y_1 + c_2 y_2$ satisifies Eq. (3), thus $y = c_1 y_1 + c_2 y_2$ is also a solution to Eq. (3).

Application of Theorem 1. In Example 1, a homogeneous second-order linear differential equation, two functions y_1 and y_2, and a pair of initial conditions are given. First verify that y_1 and y_2 are solutions of the differential equation. Then find a particular solution of the form $y = c_1 y_1 + c_2 y_2$ that satisfies the given initial conditions.

Example 1

$$y'' - 3y' + 2y = 0; \quad y_1 = e^x, \quad y_2 = e^{2x}; \quad y(0) = 1, \quad y'(0) = 7. \quad \circledast$$

ANS: If $y_1 = e^x$ then $y_1' = e^x$, $y_1'' = e^x$

Then $y_1'' - 3y_1' + 2y_1 = e^x - 3e^x + 2e^x = 0 = RHS$

Thus y_1 satisfies the given diff. eqn.

If $y_2 = e^{2x}$, then $y_2' = 2e^{2x}$, $y_2'' = 4e^{2x}$

Then $y_2'' - 3y_2' + 2y_2 = 4e^{2x} - 3\cdot 2e^{2x} + 2\cdot e^{2x} = 0 = RHS$

Thus y_2 satisfies the given eqn.

By Thm 1, we know

$y = c_1 y_1 + c_2 y_2 = \underline{c_1 e^x + c_2 e^{2x}}$ is also a solution of $\circledast$, where c_1 and c_2 are some constants.

Since $y(0)=1$, $y'(0)=7$.

$$y(0) = c_1 e^0 + c_2 e^{2\cdot 0} = \boxed{c_1 + c_2 = 1}$$

$$y'(x) = (c_1 e^x + c_2 e^{2x})' = c_1 e^x + 2c_2 e^{2x}$$

Since $y'(0)=7$, $y'(0) = c_1 e^0 + 2c_2 e^{2\cdot 0} = \boxed{c_1 + 2c_2 = 7}$

Thus c_1, c_2 satisfy

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 + 2c_2 = 7 \end{cases} \Rightarrow \begin{cases} c_1 = -5 \\ c_2 = 6 \end{cases}$$

Therefore, $y(x) = -5e^x + 6e^{2x}$ is a solution that satisfies both the diff eqn and the intial conditions $y(0)=1$, $y'(0)=7$.

Theorem 2 Existence and Uniqueness for Linear Equations

Suppose that the functions p, q, and f are continuous on the open interval I containing the point a. Then, given any two numbers b_0 and b_1, the equation

$$y'' + p(x)y' + q(x)y = f(x)$$

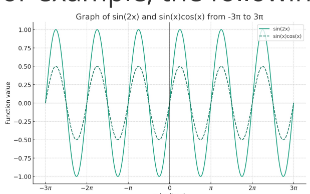
has a unique (that is, one and only one) solution on the entire interval I that satisfies the initial conditions

$$y(a) = b_0, \qquad y'(a) = b_1.$$

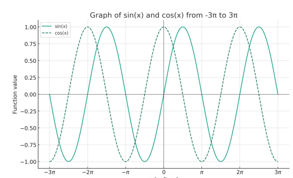
3. Linear Independence of Two Functions

Two functions defined on an open interval I are said to be **linearly independent** on I if neither is a constant multiple of the other. Two functions are said to be **linearly dependent** on an open interval if one of them is a constant multiple of the other.

For example, the following pairs of functions are linearly independent on the entire real line



$$\sin x \text{ and } \cos x$$
$$e^x \text{ and } xe^x$$
$$x+1 \text{ and } x^3$$



The functions $f(x) = \sin 2x$ and $g(x) = \sin x \cos x$ are linearly dependent.

$f(x) = 2\sin x \cos x = 2g(x)$

We can compute the **Wronskian** of two functions to determine if they are linearly independent (or dependent).

Given two functions f and g, the **Wronskian** of f and g is the determinant

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g.$$

For example,

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

and

$$W(x, 5x) = \begin{vmatrix} x & 5x \\ 1 & 5 \end{vmatrix} = 5x - 5x = 0.$$

Eg: $f(x) = x$, $g(x) = 5x$, then f and g are linearly dependent since $5f(x) = g(x)$.

Theorem 3 Wronskians of Solutions

Suppose that y_1 and y_2 are two solutions of the homogeneous second-order linear equation

$$y'' + p(x)y' + q(x)y = 0 \tag{3}$$

on an open interval I on which p and q are continuous.
(a) If y_1 and y_2 are linearly dependent, then $W(y_1, y_2) \equiv 0$ on I.
(b) If y_1 and y_2 are linearly independent, then $W(y_1, y_2) \neq 0$ at each point of I.

Theorem 4 General Solutions of Homogeneous Equations

Let y_1 and y_2 be *two linearly independent solutions* of the homogeneous equation Eq. (3)

$$y'' + p(x)y' + q(x)y = 0$$

with p and q continuous on the open interval I. If Y is any solution whatsoever of Eq. (3) on I, then there exist numbers c_1 and c_2 such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for all x in I.

Remark. We call $\{y_1, y_2\}$ a *fundamental set* of the Eq (3).

Example 2. It can be shown that $y_1 = e^{4x}$ and $y_2 = xe^{4x}$ are solutions to the differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$$

(1) Compute $W(y_1, y_2)$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{4x} & xe^{4x} \\ 4e^{4x} & \underbrace{e^{4x} + x4e^{4x}}_{=(4x+1)e^{4x}} \end{vmatrix} = (4x+1)e^{4x} \cdot e^{4x} - 4xe^{4x}e^{4x}$$

$$= (4x+1)e^{8x} - 4xe^{8x} = e^{8x} \neq 0 \text{ for } x \in (-\infty, \infty)$$

(or $x \in \mathbb{R}$)

(2) Based on the result in (1), $c_1y_1 + c_2y_2$ is the general solution to the equation on the interval $(-\infty, \infty)$.

Exercise 3. It can be shown that $y_1 = e^{2x}\sin(9x)$ and $y_2 = e^{2x}\cos(9x)$ are solutions to the differential equation $D^2y - 4Dy + 85y = 0$ on $(-\infty, \infty)$.

(1) What does the Wronskian of y_1, y_2 equal on $(-\infty, \infty)$?

(2) Is $\{y_1, y_2\}$ a fundamental set for $D^2y - 4Dy + 85y = 0$ on $(-\infty, \infty)$?

Solution.

(1)

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= \begin{vmatrix} e^{2x}\sin(9x) & e^{2x}\cos(9x) \\ \left(e^{2x}\sin(9x)\right)' & \left(e^{2x}\cos(9x)\right)' \end{vmatrix} \\
&= \begin{vmatrix} e^{2x}\sin(9x) & e^{2x}\cos(9x) \\ (2\sin(9x) + 9\cos(9x))e^{2x} & (-9\sin(9x) + 2\cos(9x))e^{2x} \end{vmatrix} \\
&= \left(e^{2x}\sin(9x)\right) \cdot \left((-9\sin(9x) + 2\cos(9x))e^{2x}\right) - \left(e^{2x}\cos(9x)\right)\left((2\sin(9x) + 9\cos(9x))e^{2x}\right) \\
&= -9e^{4x}(\sin^2(9x) + \cos^2(9x)) \\
&= -9e^{4x}
\end{aligned}$$

(2) Yes, since $W\left(y_1, y_2\right) \neq 0$ on $\left(-\infty, \infty\right)$

Exercise 4. For the differential equation $y'' + 4y' + 13y = 0$, a general solution is of the form $y = e^{-2x}\left(C_1 \sin 3x + C_2 \cos 3x\right)$, where C_1 and C_2 are arbitrary constants. Applying the initial conditions $y(0) = -2$ and $y'(0) = 10$, find the specific solution.

Solution.
Applying the initial condition $y(0) = -2$, we get,

$$y(0) = C_2 = -2.$$

To apply the initial condition $y'(0) = 10$, first find $y'(x)$.

$$y'(x) = e^{-2x}\left(3C_1 \cos 3x - 3C_2 \sin 3x\right) - 2e^{-2x}\left(C_1 \sin 3x + C_2 \cos 3x\right).$$

Therefore,

$$y'(0) = 3C_1 - 2C_2 = 10.$$

This leads to the following two equations in terms of C_1 and C_2,

$$\begin{aligned} C_2 &= -2 \\ 3C_1 - 2C_2 &= 10 \end{aligned}$$

Solving this system leads to $C_1 = 2$ and $C_2 = -2$. Therefore the specific solution is,

$$y = e^{-2x}(2\sin(3x) - 2\cos(3x))$$